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ONE-DIMENSIONAL MODELS FOR TRANSIENT GAS–LIQUID FLOWS IN DUCTS

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Abstract—One-dimensional models are presented for simple, separated (stratified, annular) and mixed (bubble, droplet) gas–liquid flows. Specific attention is focused on equations for (i) interface and wall transfers of heat and momentum; (ii) the difference between phase and interface pressure; (iii) interface pressure; and (iv) space-time distribution parameters. Equations appropriate to simple separated and mixed flows are investigated to determine their effect on the propagation velocities of acoustic and interfacial waves. Solutions to two simple numerical problems, obtained by using these equations, are also discussed. Finally, recommendations are made for a more general transient model.

INTRODUCTION

Mathematical models for transient gas–liquid flow are usually derived starting from the local instantaneous differential conservation laws and interface jump conditions (Standart 1964). Because the general three-dimensional problem with moving interfaces is too formidable to solve, the universal approach is to introduce time or statistical averaging and space averaging to remove the need to treat the interfaces explicitly (Vernier & Delhaye 1968; Ishii 1975; Delhaye 1977). Models of varying sophistication result from specific choices for the averaging operators and assumptions about the local relationships between phase variables (e.g. equality of phase velocities and/or temperatures). In this paper, we focus attention on one-dimensional models for transient boiling flows that allow for differences between the velocities and temperatures of the gas and liquid phases.

Of particular interest are vapour–liquid flows produced by rapid depressurization, due to discharge through a break, of an initially subcooled liquid flowing in a network of pipes. In general, the flow structure may be idealized as follows:

(i) *Mixed flows* in which either gas or liquid packets (bubbles/drops) are dispersed in a liquid or vapour matrix. These flows are characterized by strong coupling between the phases, due to rapid interphase transfers of mass, momentum and energy, so that large temperature and velocity differences cannot be sustained.

(ii) *Separated flows* in which the gas and liquid are separated by a continuous interface. These flows are characterized by weak coupling between the phases (i.e. slow interphase transfers) so that relatively large differences in temperature and velocity can be sustained between the phases. Flow regimes covered by this classification include vertical and horizontal annular flow and horizontal stratified flow.

(iii) *Transition flows* that represent transient states between (i) and (ii). These flows are characterized by geometrically complex interfaces and interphase coupling that is intermediate between that for the mixed and separated flows (e.g. the churn flow in which neither phase is continuous, or annular flow with liquid droplets entrained in the vapour core).

During a transient, mixed, separated and transition flows will be present at the same time but at different spatial locations. We seek a consistent description of the various flow regimes within a single mathematical model. In choosing this approach, it is recognized that the description of a particular flow regime may not be as complete as that provided by a specialized model for that flow regime.

Many models that incorporate the effects of unequal phase velocities and temperatures, have been proposed (Hughes *et al.* 1976; Stuhmiller 1977; Banerjee *et al.* 1978; Mathers *et al.* 1978; Agee *et al.* 1978; Rousseau & Ferch 1979) but none has received general acceptance. All of these models represent attempts to get appropriate physical behaviour for specific flow regimes while retaining the hyperbolic character of the flow equations. None of these models is sufficiently general to treat transient boiling flows.

We have adopted as our starting point the instantaneous, area-averaged form of the conservation laws and interface jump conditions derived by Delhaye (1977). Depending on flow structure, these equations are either ensemble averaged or time averaged to account for random fluctuations. This results in a system of three partial differential equations for mass, momentum and energy conservation in each phase and three jump conditions relating the average mass, momentum and energy transfers across the interfaces. To close the system, additional equations (or simplifying assumptions about the flow structure) are required for:

(i) The thermodynamic state of each phase.

(ii) Transfers of mass, momentum and energy from each phase to the interface and pipe wall.

(iii) Interface properties as functions of the space-time averaged flow variables.

(iv) Space-time distribution parameters that represent the difference between the average of products and product of averages for various mass, momentum and energy fluxes.

Although all of the terms required to close the equations are precisely defined in a mathematical sense, sufficiently detailed information about the flow structure is not usually available to determine their relationship to the space-time averaged flow variables. It is usual to neglect arbitrarily some of these terms (e.g. the space-time distribution parameters) and use simplified physical arguments, appropriate to the flow regime of interest, to determine the functional form of the remaining terms. The internal consistency of the models developed in this way is open to question but cannot be checked because suitable experimental data are not available. In addition, the functional form of the closure terms is often defined to make the resultant flow equations hyperbolic. Whether the equations should be hyperbolic or not remains unresolved.

In the following sections, the fundamental equations for one-dimensional gas–liquid flow in a duct are presented and discussed. Specific attention is focused on the equations required to close the system. Equations are proposed for the interface and wall transfer terms which contain local instantaneous values of the primary flow variables, but not their derivatives. A single interfacial pressure is defined, consistent with the assumption that surface tension effects are negligible. This focuses attention on the definition of functions for the difference between phase and interface pressure, and space-time distribution parameters. Various functions are investigated to determine their effect on the propagation velocities. Solutions to simple standard problems, obtained using these functions are also presented. Finally recommendations are made for a general flow-boiling model.

FUNDAMENTAL EQUATIONS FOR-DIMENSIONAL GAS-LIQUID FLOW

Using the methodology of Delhaye (1977), the area- and time/ensemble-averaged mass, momentum and energy conservation laws for phase k($k = 1$ for gas; $k = 2$ for liquid) flowing in a constant area duct are respectively:†

$$\frac{\partial}{\partial t}\alpha_k\rho_k + \frac{\partial}{\partial x}\alpha_k\rho_k u_k = m_{ki} - \frac{\partial \Delta_{lk}}{\partial t} - \frac{\partial \Delta_{2k}}{\partial x} \qquad [1]$$

†We have assumed that the terms $(\partial/\partial t)\ \alpha_k\langle \underline{n}_x \cdot (\underline{n}_k \cdot \underline{\underline{\tau}}_k)\rangle$ and $(\partial/\partial x)\ \alpha_k\langle \underline{n}_k \cdot (\underline{q}_k - \underline{\underline{\tau}}_k \cdot \underline{v}_k)\rangle$ in the momentum and energy equations respectively are negligible. A no-slip condition at the wall ($u_{kw} = 0$) has also been assumed throughout.

$$\frac{\partial}{\partial t}\alpha_k\rho_k u_k + \frac{\partial}{\partial x}\alpha_k\rho_k u_k^2 + \alpha_k\frac{\partial p_k}{\partial x} + (p_k - p_i)\frac{\partial \alpha_k}{\partial x}$$

$$= \tau_{ki} - m_{ki}u_{ki} - p'_{ki} + \tau_{kw} + \alpha_k\rho_k b_x - \frac{\partial \Delta_{2k}}{\partial t} - \frac{\partial}{\partial x}(\Delta_{3k} + \Delta_{4k}) \quad [2]$$

$$\frac{\partial}{\partial t}\alpha_k\rho_k(h_k + u_k^2/2) + \frac{\partial}{\partial x}\alpha_k\rho_k u_k(h_k + u_k^2/2) - \alpha_k\frac{\partial P_k}{\partial t} - (p_k - p_i)\frac{\partial \alpha_k}{\partial t}$$

$$= m_{ki}(h_i + u'^2_{ki}/2) - q_{ki} + \tau'_{ki}u_{ki} - p''_{ki} - q_{kw} + \alpha_k\rho_k u_k b_x + \frac{\partial}{\partial t}(\Delta_{4k} - \Delta_{5k}) - \frac{\partial \Delta_{6k}}{\partial x} \quad [3]$$

where α_k, ρ_k, u_k, p_k and h_k denote respectively area fraction, density, x-component of velocity, pressure and enthalpy of phase k, p_i is an averaged interface pressure, and b_x is the x-component of an externally applied body force. Terms appearing on the r.h.s's are defined below. All of the dependent variables, with the exception of α_k which is only time averaged, are area and time (or ensemble) averaged as follows:

$$f_k \equiv \begin{cases} \int_T \left\{ \int_{A_k} \mathrm{f}_k \,\mathrm{d}A/A_k \right\} \mathrm{d}t/T \\ \sum_N \left\{ \int_{A_k} f_k \,\mathrm{d}A/A_k \right\}/N \end{cases} \quad [4]$$

where A_k is the instantaneous cross-section area of phase k, T is a time interval which is large in comparison to the high frequency "turbulent" fluctuations but small in comparison to gross flow fluctuations, and N is the number of samples.

The rate of transfer of phase-k mass per unit volume from the interface is defined as:

$$m_{ki} \equiv \overline{\langle\langle \rho_{k} \underset{=}{n}_{k} \cdot (\underset{=}{v}_k - \underset{=}{v}_i)\rangle\rangle_i}/A = \overline{\langle\langle m_k \rangle\rangle_i}/A$$

where $\underline{n}_k$ is the outwardly directed unit normal vector at the interface, A is the duct flow area, $\underline{v}_k$ and $\underline{v}_i$ are respectively phase-k and interface velocity vectors, the operator $\langle\langle\;\rangle\rangle_i$ denotes the line integral taken about the interface $\int_c \mathrm{d}c/\sqrt{[1-(\underline{n}_x \cdot \underline{n}_k)^2]_i}$ where $\underline{n}_x$ is the unit vector in the x-direction, and the overbar denotes either time average $\int_T \mathrm{d}t/T$ or ensemble average $1/N\sum_N$. The other interface variables are defined as:

$$\tau_{ki} \equiv \overline{\langle\langle \underline{n}_x \cdot (\underline{n}_k \cdot \tau_k)\rangle\rangle_i}/A; \qquad \tau_{kw} \equiv \overline{\langle\langle \underline{n}_x \cdot (\underline{n}_k \cdot \underline{\underline{\tau}}_k)\rangle\rangle_w}/A$$

$$u_{ki} \equiv \overline{\langle\langle m_k u_k \rangle\rangle_i}/Am_{ki}; \qquad u'^2_{ki} \equiv \overline{\langle\langle m_k v_k^2 \rangle\rangle_i}/Am_{ki}$$

$$\tau'_{ki} \equiv \overline{\langle\langle \underline{n}_k \cdot (\underline{v}_k \cdot \underline{\underline{\tau}}_k)\rangle\rangle_i}/Au_{ki}$$

$$q_{ki} \equiv \overline{\langle\langle \underline{n}_k \cdot \underline{q}_k \rangle\rangle_i}/A; \qquad q_{kw} \equiv \overline{\langle\langle \underline{n}_k \cdot \underline{q}_k \rangle\rangle_w}/A$$

$$h_{ki} \equiv \overline{\langle\langle m_k h_k \rangle\rangle_i}/Am_{ki}$$

where $\underline{\underline{\tau}}_k$ is the viscous stress tensor, $\underline{q}_k$ is the heat flux vector and $\langle\langle\,\rangle\rangle_w$ denotes the line integral taken around that portion of the wall in contact with phase-k. The interface pressure terms in the momentum and energy equations are split into two components as follows:

$$\overline{\langle\langle \underline{n}_x \cdot \underline{n}_k p_k \rangle\rangle_i}/A \equiv p'_{ki} + p_i\overline{\langle\langle \underline{n}_x \cdot \underline{n}_k \rangle\rangle_i}/A$$

$$\overline{\langle\langle \underline{n}_k \cdot \underline{v}_k p_k \rangle\rangle_i}/A \equiv p''_{ki} + p_i\overline{\langle\langle \underline{n}_k \cdot \underline{v}_k \rangle\rangle_i}/A. \quad [5]$$

Equations [5] define the terms p'_{ki} and p''_{ki}. The particular form is chosen for convenience, because the second terms on the r.h.s's can be transformed using the special forms of the Gauss and Leibniz theorems derived by Delhaye (1977) to obtain:

$$\langle\langle \underline{n}_x \cdot \underline{n}_k p_k \rangle\rangle_i / A \equiv p'_{ki} - p_i \frac{\partial \alpha_k}{\partial x}$$

$$\langle\langle \underline{n}_k \cdot \underline{v}_k p_k \rangle\rangle_i / A \equiv p''_{ki} + p_i \frac{\partial \alpha_k}{\partial t}. \qquad [6]$$

It is appropriate to note that the interface terms containing m_k and τ_k could be transformed in a similar manner. However, this would tend to obscure their physical significance. Our choice is motivated by the desire to define an interface pressure which will prove convenient in the description of separated flows.

The space-time distribution parameters Δ_{jk} ($j = 1, 2 \ldots, 6$) are defined as follows:

$$\begin{aligned}
\Delta_{1k} &\equiv \overline{\alpha_k \langle \rho_k \rangle} - \alpha_k \rho_k \\
\Delta_{2k} &\equiv \overline{\alpha_k \langle \rho_k u_k \rangle} - \alpha_k \rho_k u_k \\
\Delta_{3k} &\equiv \overline{\alpha_k \langle \rho_k u_k^2 \rangle} - \alpha_k \rho_k u_k^2 \\
\Delta_{4k} &\equiv \overline{\alpha_k \langle p_k \rangle} - \alpha_k p_k \\
\Delta_{5k} &\equiv \overline{\alpha_k \langle \rho_k (h_k + v_k^2/2) \rangle} - \alpha_k \rho_k (h_k + u_k^2/2) \\
\Delta_{6k} &\equiv \overline{\alpha_k \langle \rho_k u_k (h_k + v_k^2/2) \rangle} - \alpha_k \rho_k u_k (h_k + u_k^2/2).
\end{aligned} \qquad [7]$$

These distribution parameters are usually assumed to be constant, so that the terms involving their derivatives are zero. The assumption is made for mathematical convenience rather than for physical reasons. The parameters represent the macroscopic effects of the flow microstructure that has been removed by the averaging process. It will be seen later that, because the Δ_{jk} appear as derivatives, they will affect the propagation velocities of small interface and pressure disturbances.

The area and time/ensemble-averaged jump conditions for mass, momentum and energy transfer across the interface are:

$$\sum_k m_{ki} = 0 \qquad [8]$$

$$\sum_k \{\tau_{ki} - m_{ki} u_{ki} - p'_{ki}\} = 0 \qquad [9]$$

$$\sum_k \{m_{ki}(h_{ki} + u'^2_{ki}/2) + q_{ki} - \tau'_{ki} u_{ki} + p''_{ki}\} = 0. \qquad [10]$$

These equations follow from the local instantaneous conservation equations, integrated over a volume containing only the immediate neighbourhood of the interface (Vernier & Delhaye 1968). The contribution of surface tension has been assumed negligible. Also, it is at this point that the form chosen for [5] becomes useful; by assuming the same proportionality constant p_i for both phases, we have cancelled the derivatives of α_k that would otherwise appear in [9] and [10].

The conservation equations [1]–[3] and jump conditions [8]–[10] contain 45 unknowns. Additional equations are provided by noting that $\Sigma\alpha_k = 1$, assuming that the space-time-averaged phase properties ρ_k, h_k and p_k are related through the thermodynamic state equations such as $\rho_k = \rho_k(h_k, p_k)$ and making the simplifying assumptions that $u_{ki} = u'_{ki}$ and $\tau_{ki} = \tau'_{ki}$. To close the system, we need equations for each of the following variables as functions of the primary flow variables α_k, u_k, h_k and p_k:

(i) Interface variables u_{ki}, h_{ki}, p'_{ki} and p''_{ki}.

(ii) Phase-to-interface transfers τ_{1i}, q_{1i} and q_{2i}. The remaining transfer terms τ_{2i}, m_{1i} and m_{2i} are then defined by the jump conditions [8]–[10].

(iii) Phase-to-wall transfers τ_{kw} and q_{kw}.

(iv) Phase-to-interface pressure differences $(p_k - p_i)$.

(v) Space-time distribution parameters Δ_{jk}, $j = 1, 2, \ldots 6$.

The choice of specific equations for terms (i)–(v) depends on the flow structure and some examples are discussed in the following sections.

Before proceeding, it is convenient to note that the conservation equations [1]–[3] can be algebraically manipulated into the following matrix form:

$$\underset{=}{A}\frac{\partial \underset{-}{U}}{\partial t} + \underset{=}{B}\frac{\partial \underset{-}{U}}{\partial x} = \underset{-}{C}$$

where $\underset{=}{A}$ and $\underset{=}{B}$ are square matrices of coefficients which are functions of the dependent flow variables, $\underset{-}{U}$ is a vector of dependent flow variables, and $\underset{-}{C}$ is a vector containing allowances for interfacial and wall transfers of mass, momentum and energy. The nature of this system of equations may be determined from the eigenvalues of the characteristic determinant $\|B - \lambda A\|$. The eigenvalues λ represent the velocities of propagation of small-amplitude, short-wavelength perturbations (Whitham 1974). For long-wavelength disturbances, dispersive effects and the source terms contained in $\underset{-}{C}$ become important, while at large amplitudes non-linear wave interactions dominate. However, for small disturbances such as weak acoustic and interfacial waves, the characteristic analysis is adequate. If all of the eigenvalues λ are real, the flow equations are hyperbolic and stable against such small disturbances. The propagation velocities obtained in this way will prove useful in assessing the physical viability of the flow models discussed below.

SEPARATED FLOWS

For stratified and annular gas–liquid flows with small amplitude interfacial waves, it seems reasonable to assume that:

(i) The interface pressure is constant at any cross-section and thus p'_{ki} and p''_{ki} are zero.

(ii) The space-time distribution parameters Δ_{jk} are constant so that

$$\frac{\partial \Delta_{jk}}{\partial t} \text{ and } \frac{\partial \Delta_{jk}}{\partial x} \text{ are zero.}$$

(iii) The interface velocity is approximately equal to the average liquid velocity, i.e. $u_{ki} = u_2$. Note that this imposes the additional restriction that m_{ki} must be small.

In addition, we assume that the gas and liquid adjacent to the interface are at the local saturation condition corresponding to p_i. It follows that $h_{ki} = h_k(p_i)$; see the analysis of Ardron & Duffey (1978) for a justification of this assumption.

The required interface heat and momentum transfer terms are defined as follows:

$$\begin{aligned} q_{ki} &= \lambda_{ki} c_i \{T(p_i) - T_k\}/A \\ \tau_{1i} &= 1/2 f_i c_i \rho_i |u_1 - u_2|(u_1 - u_2)/A \end{aligned} \qquad [11]$$

where λ_{ki} is an appropriate heat transfer coefficient, f_i is the interface friction factor, c_i is the length of the interface in the cross-section plane, and $T(p_i)$ and T_k denote respectively the interface and phase-k temperature. Similarly the wall transfer terms are:

$$q_{kw} = \lambda_{kw} c_{kw}(T_w - T_k)/A$$
$$\tau_{kw} = 1/2 f_{kw} c_{kw} \rho_k u_k^2 / A \qquad [12]$$

where the subscript kw denotes that portion of the wall in contact with phase k (note that $c_{1w} = 0$ and $c_{2w} = \pi d$ for annular flow in a duct of diameter d).

To complete the set of closure equations, we need only define $(p_k - p_i)$ and an equation of the following form is proposed:

$$(p_k - p_i) = \alpha_k \rho_k \xi_k(u_1, u_2) \quad k = 1, 2 \qquad [13]$$

where ξ_k must be defined consistently with the physical situation of interest. Before examining specific functions for ξ_k, it is convenient to remove the need to consider the energy equations by assuming a constant enthalpy flow such that $\rho_k = \rho_k(p_k)$. The corresponding phase-k mass and momentum equations are respectively:

$$\frac{D_k \alpha_k}{Dt} + \frac{\alpha_k}{\rho_k a_k^2}\frac{D_k p_k}{Dt} + \alpha_k \frac{\partial u_k}{\partial x} = \frac{m_{ki}}{\rho_k} \qquad [14]$$

$$\frac{D_k u_k}{Dt} + \frac{1}{\rho_k}\frac{\partial p_k}{\partial x} + \frac{p_k - p_i}{\alpha_k \rho_k}\frac{\partial \alpha_k}{\partial x} = \{\tau_{ki} + \tau_{kw} - m_{ki}(u_{ki} + u_k)\}/\alpha_k \rho_k \qquad [15]$$

where $a_k \equiv \sqrt{(\partial p_k / \partial \rho_k)}$ is the speed of sound in phase k and $D_k/Dt = \partial/\partial t + u_k\, \partial/\partial t$. From [13] it follows that

$$p_2 = p_1 + \alpha_2 \rho_2 \xi_2 - \alpha_1 \rho_1 \xi_1$$

and hence

$$\frac{\partial p_2}{\partial \eta} = \beta_p \frac{\partial p_1}{\partial \eta} + \beta_\alpha \frac{\partial \alpha_1}{\partial \eta} + \beta_{u1}\frac{\partial u_1}{\partial \eta} + \beta_{u2}\frac{\partial u_2}{\partial \eta} \qquad [16]$$

substitution of [16] (with $\eta = t, x$) into [14] and [15] allows the characteristic determinant $\|B - \lambda A\|$ to be written as

$$\left\| \begin{array}{cccc} (u_1 - \lambda) & \dfrac{\alpha_1}{\rho_1 a_1^2}(u_1 - \lambda) & \alpha_1 & 0 \\ \hline \left[\dfrac{\alpha_2 \beta_2}{\rho_2 a_2^2} - 1\right](u_2 - \lambda) & \dfrac{\alpha_2 \beta_p}{\rho_2 a_2^2}(u_2 - \lambda) & \dfrac{\alpha_2 \beta_{u1}}{\rho_2 a_2^2}(u_2 - \lambda) & \dfrac{\alpha_2 \beta_{u_2}}{\rho_2 a_2^2}(u_2 - \lambda) + \alpha_2 \\ \hline \xi_1 & \dfrac{1}{\rho_1} & (u_1 - \lambda) & 0 \\ \hline \dfrac{\beta_\alpha}{\rho_2} - \xi_2 & \dfrac{\beta_p}{\rho_2} & \dfrac{\beta_{u_1}}{\rho_2} & \dfrac{\beta_{u_2}}{\rho_2} + (u_2 - \lambda) \end{array} \right\| = 0 \qquad [17]$$

where $\beta_p = (1 - \alpha_1 \xi_1 / a_1^2)/(1 - \alpha_2 \xi_2 / a_2^2)$; $\beta_\alpha = (\rho_1 \xi_1 + \rho_2 \xi_2)/(1 - \alpha_2 \xi_2 / a_2^2)$; $\beta_{u_1} = (\alpha_2 \rho_2 \partial \xi_2 / \partial u_1 - \alpha_1 \rho_1 \partial \xi_1 / \partial u_1)/(1 - \alpha_2 \xi_2 / a_2^2)$; and $\beta_{u_2} = (\alpha_2 \rho_2 \partial \xi_2 / \partial u_2 - \alpha_1 \rho_1 \partial \xi_1 / \partial u_2)/(1 - \alpha_2 \xi_2 / a_2^2)$.

As will be seen below, the eigenvalues of [17] are the propagation velocities of acoustic and interfacial waves.

Consider, as a special case, a stratified gas–liquid flow in a horizontal rectangular duct of height H. As a simple model for $(p_k - p_i)$, we assume that the phase pressures vary linearly over the cross-section. A transverse momentum balance, neglecting all effects except the gravitational force, yields the following relationships:

$$p_k - p_i = (-1)^k \alpha_k \rho_k g H/2 \qquad [18]$$

so that $\xi_1 = -gH/2 = -\xi_2$. Making use of the fact that $gH \ll a^2{}_k$ we can write

$$\beta_p = 1;\ \beta_\alpha = -(\rho_2 - \rho_1) gH/2;\ \beta_{u_1} = 0;\ \beta_{u_2} = 0.$$

As shown by Rousseau & Ferch (1979), the above choice of closure equation for $(p_k - p_i)$ gives the approximate eigenvalues

$$\lambda = u_s \pm v_s,\ u_s^* \pm a_s^* \qquad [19]$$

where $u_s = (\alpha_2\rho_1 u_1 + \alpha_1\rho_2 u_2)/\rho^*$; $u_s^* = (\alpha_2\rho_1 u_2 + \alpha_1\rho_2 u_1)/\rho^*$; $v_s^2 = \alpha_1\alpha_2\{(\rho_2 - \rho_1)gH - \rho_1\rho_2(u_1 - u_2)^2/\rho^*\}/\rho^*$; $a_s^{*2} = \rho^*/(\alpha_2\rho_1/a_2{}^2 + \alpha_1\rho_2/a_1{}^2$; $\rho^* = \alpha_2\rho_1 + \alpha_1\rho_2$.

The pair of eigenvalues $u_s^* \pm a_s^*$ correspond to the propagation velocities of acoustic waves. The mixture speed of sound a_s^* is commonly referred to as the "stratified flow" speed of sound (Wallis 1969). It can be seen that $a_1 < a_s^* < a_2$ and, except at very small gas volume fractions, a_s^* is approximately equal to a_1. This appears to be in good agreement with experimental data for stratified and annular flows (Henry 1971). It is also in agreement with the speed of sound derived by Morioka & Matsui (1975) for a two-dimensional stratified flow including transverse velocity components. The translation velocity associated with these acoustic waves, u_s^*, is approximately equal to u_1. Hence the propagation of sound waves is dominated by the gas phase. It is of interest to compare a_s^*, in the limit as $(u_1 - u_2)$ approaches zero, with the speed of sound corresponding to the equal-velocity-unequal-temperature (EVUT) flow model (Hancox *et al.* 1978; Ferch 1979). As can be seen from figure 1, a_s^* differs markedly from the EVUT sound speed. We attribute this difference to the fact that both $(u_1 - u_2)$ and its derivatives are set equal to zero in the derivation of the EVUT model, while only $(u_1 - u_2)$ is zero when the limiting

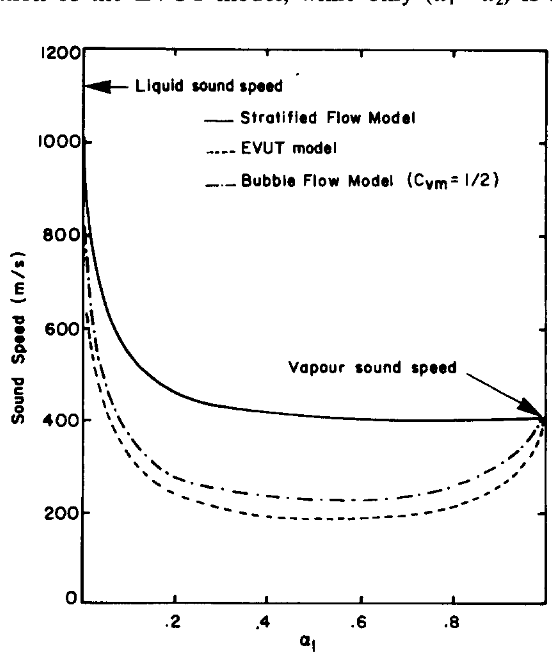

Figure 1. Comparison of sound speed for three models with $u_1 = u_2$.

value of a_s^* is determined. This underscores the need to apply limiting processes carefully when comparing different flow models.

The other pair of eigenvalues $u_s \pm v_s$ correspond to the propagation velocities of interfacial waves. In this case, u_s is appromixately equal to u_2, and as expected, the liquid phase dominates. We note that v_s is real only when

$$(u_1 - u_2)^2 < \rho^*(\rho_2 - \rho_1)gH/\rho_1\rho_2. \quad [20]$$

When [20] is not satisfied, the dispersion relation for small-amplitude sinusoidal perturbations, which is identical to [17] for short wavelengths, indicates that short-wavelength interfacial waves will grow exponentially in amplitude. We interpret this as representing a transition to large, but finite, amplitude interfacial waves. Therefore the conditions for which our stratified flow equations cease to be hyperbolic correspond to the conditions for which our simplifying assumptions ($p'_{ki} = 0$; $p''_{ki} = 0$; Δ_{jk} = constant) are no longer consistent with the flow structure. To improve the description of the flow structure, we must derive more representative closure equations. We will return to this matter later.

For annular flow, a simple model for $(p_k - p_i)$ may be obtained using reasoning similar to that for flow over a wavy wall. This suggests that

$$p_k - p_i = \alpha_k\rho_k C_{pk}(u_i - u_2)^2/2 \quad k = 1, 2 \quad [21]$$

where C_{pk} is a dynamic pressure coefficient. In terms of the general equations [13], we see that

$$\xi_k = C_{pk}(u_1 - u_2)^2/2$$

and for the special case $C_{pk}(u_1 - u_2)^2 \ll a^2{}_k$ (consistent with the assumption of small amplitude interfacial waves)

$$\beta_p = 1;\ \beta_\alpha = -C'_p(u_1 - u_2)^2/2;\ \beta_{u_1} = C^*_p(u_1 - u_2);\ \beta_{u_2} = -C^*_p(u_1 - u_2)$$

where $C'_p = \rho_1 C_{p1} + \rho_2 C_{p2}$ and $C^*_p = \alpha_1\rho_2 C_{p2} - \alpha_1\rho_1 C_{p1}$.

Substitution for the above terms in determinant [17] and calculation of the eigenvalues yields acoustic propagation velocities that are nearly identical to those for the stratified flow case, i.e. $a^* \cong a_1$. The propagation velocities of interfacial waves are

$$\lambda = u_a \pm v_a \quad [22]$$

where $u_a = u_s - C^*_p(u_1 - u_2)/2\rho^*$ and

$$v_a{}^2 = [\alpha_1\alpha_2\{\rho^* C'_p + (\rho_2 - \rho_1)C^*_p - \rho_1\rho_2\} + (C^*_p/2)^2](u_1 - u_2)^2/\rho^{*2}.$$

We note that v_a is real only when

$$\frac{C^{*2}_p + 4\alpha_1\alpha_2[\rho^* C'_p + (\rho_2 - \rho_1)C^*_p]}{4\alpha_1\alpha_2\rho_1\rho_2} > 1. \quad [23]$$

Equation [23] imposes a constraint on the smallest values allowed for C_{pk}, i.e. minimum values for a stable annular film to exist. For the special case $C_{p2} = -C_{p1} = C_p$, [23] yields the approximate condition ($\rho_1 \ll \rho_2$)

$$C_p > \rho_1/\rho_2.$$

This annular flow model does not explicitly impose an upper bound on C_p although it is clear that the onset of either entrainment or large amplitude interfacial waves is inconsistent with the assumptions made in the derivation of the separated flow equations. Here we must rely on external criteria, based on physical arguments, for flow regime transition such as those presented by Dukler (1978).

MIXED FLOWS

Mixed flows are characterized by much stronger coupling between the gas and liquid phases than exists in separated flows. As a special case we will consider a flow of bubbles (or droplets) dispersed in a liquid (or gas) matrix. The following assumptions appear consistent with experimental observations:

(i) The phase pressures are equal at any cross-section, i.e. $p_1 = p_2 = p$. Since we will assume that $p \neq p_i$, it would be more appropriate to set the pressure of the dispersed phase equal to the interface pressure. However, because p_i will not be very different from p, the error in our assumption will be small.

(ii) The gas and liquid adjacent to the interface are at the local saturation condition corresponding to the average pressure p so that $h_{ki} = h_{ki}(p)$.

(iii) The interface velocity is equal to the average velocity of the dispersed phase, i.e. $u_{ki} = u_q$ for a bubble flow.

The interface heat transfer terms are defined as

$$q_{ki} = \lambda_{ki} c_i (T(p) - T_k)/A \qquad [24]$$

where $c_i = 3\alpha_d/r_d$, and α_d and r_d denote respectively the volume fraction and average radius of the dispersed phase. Heat transfer from the wall to the continuous phase is

$$q_c = 4\lambda_{cw}(T_w - T_c)/d \qquad [25]$$

where the subscript c denotes the continuous phase. Heat transfer from the wall to the dispersed phase is assumed to be zero. Similarly, the wall shear stress experienced by the continuous phase is

$$\tau_c = 2f\rho_c u_c^2/d^2. \qquad [26]$$

To complete the set of closure equations, we need equations for p'_{ki}, p''_{ki}, Δ_{ki}, τ_{di} and $(p - p_i)$. For an idealized bubble or droplet flow it should be possible to define unique equations for each of these terms but this has not been done. The usual approach is to neglect some of the terms arbitrarily (e.g. $\Delta_{jk} = 0$) and then use simple physical arguments for those that remain. This leads, for example, to the following functions for $(p - p_i)$ and τ_{di} (Stuhmiller 1977):

$$(p - p_i) = C_p \rho_c (u_d - u_c)^2 \qquad [27]$$

$$\tau_{di} = C_D c_i \rho_c |u_d - u_c|(u_d - u_c)/A \qquad [28]$$

where C_p and C_D are respectively dynamic pressure and drag coefficients. An equation for the interfacial pressure component p'_{ki} is then derived from the classical theory for forces acting on an accelerating sphere and Stuhmiller writes

$$p'_{di} = -p'_{ci} = -\alpha_d \rho_c C_{vm} \left\{ \frac{\partial}{\partial t}(u_c - u_d) + u_d \frac{\partial}{\partial x}(u_c - u_d) \right\} \qquad [29]$$

where C_{vm} is the virtual mass coefficient (1/2 for non-interacting spheres). Recent work by

Drew & Lahey (1979) suggests the following more general function

$$p'_{di} = -p'_{ci} = -\alpha_d\rho_c C_{vm}\left[\frac{\partial u_c}{\partial t} + \{u_d - (1-\lambda_d)(u_c - u_d)\}\frac{\partial u_c}{\partial x}\right]$$
$$+\alpha_d\rho_c C_{vm}\left[\frac{\partial u_d}{\partial t} + \{u_c - (1-\lambda_d)(u_c - u_d)\}\frac{\partial u_d}{\partial x}\right] \quad [30]$$

where λ_d is a coefficient that must be determined from experiments (they note that as $\alpha_d \to 0$, $\lambda_d \to 2$).

To simplify the determination of the propagation velocities corresponding to the above choice of closure equations, we will again consider a constant enthlapy case. The resulting characteristic determinant with [30] is

$$\begin{Vmatrix} (u_d-\lambda) & \dfrac{\alpha_d}{\rho_d a_d^2}(u_d-\lambda) & \alpha_d & 0 \\ -(u_c-\lambda) & \dfrac{\alpha_c}{\rho_c a_c^2}(u_c-\lambda) & 0 & \alpha_c \\ \dfrac{p-p_i}{\alpha_d\rho_d} & \dfrac{1}{\rho_d} & (u_d-\lambda)+c'_d(u_{vd}-\lambda) & -c'_d(u_{vc}-\lambda) \\ -\dfrac{p-p_i}{\alpha_c\rho_c} & \dfrac{1}{\rho_c} & -c'_c(u_{vd}-\lambda) & (u_c-\lambda)+c'_c(u_{vc}-\lambda) \end{Vmatrix} = 0 \quad [31]$$

where u_{vd} and u_{vc} are the convective velocities of the virtual mass associated with the dispersed and continuous phases respectively and $c'_k \equiv \alpha_d\rho_c C_{vm}/\alpha_k\rho_k$. In general, the eigenvalues of determinant [31] cannot be obtained in a simple analytic form. However, we note that they have the general form

$$\lambda = u_m \pm v_m, \quad u_m^* \pm a_m^*$$

where $u_m \pm v_m$ and $u_m^* \pm a_m^*$ correspond to the propagation velocities of interfacial and acoustic waves respectively. For our present purposes it is sufficient to consider only the limiting cases of incompressible flow and of u_d approaching u_c.

For an incompressible flow (i.e. $\alpha_k(u_k - \lambda)/\rho_k \ll a_k^2$) the propagation velocities of interfacial waves are defined as follows:

$$u_m = \{\alpha_c\rho_d u_d + \alpha_d\rho_c u_c + [2u_d - \alpha_d(u_d - u_c)]C^*/2\}/\rho^*$$
$$v_m^2 = \{(\rho^* + C^*)C_p\rho_c + \alpha_d^2 C^{*2}/4 - \alpha_c\alpha_d\rho_c(\rho_d + C^*)\}\Delta u^2/\rho^{*2} \quad [32]$$

where $C^* = \rho_c C_{vm}/\alpha_c$. The condition for v_m to be real, i.e. for the bubble/droplet flow to be stable, is

$$\frac{4C_p\rho_c(\rho^* + C^*) + \alpha_d^2 C^{*2}}{4\alpha_c\alpha_d\rho_c(\rho_d + C^*)} > 1 \quad [33]$$

and this establishes the smallest values allowed for the dynamic pressure and virtual mass coefficients. We see that for $C_p = 0$, C_{vm} must be $> 4\alpha^2{}_c/\alpha_d$ and for $C_{vm} = 0$, C_p must be $> \alpha_c\alpha_d\rho_d/\rho^*$.

In the limit as u_d approaches u_c, the acoustic propagation velocity a_m^* is given by

$$a_m^{*2} = a_s^{*2}(1 + C_{vm}\rho_c/\alpha_c\rho^*)/(1 + C_{vm}\rho_c/\rho_d + C_{vm}\alpha_d/\alpha_c). \quad [34]$$

We note that if $C_{vm} = 0$, $a_m^* = a_s^*$, while in the limit as C_{vm} becomes large, a_m^* approaches the speed of sound corresponding to the EVUT model. Figure 1 shows a comparison between a_m^* ($C_{vm} = 1/2$) and sound speeds corresponding to the EVUT and separated flow models. Ardron & Duffey (1978) have done a detailed analysis of the dispersion equation corresponding to the case $u_{vc} = u_{vd} = u_d$, $C_p = 0$ and $C_{vm} = (1 + 2\alpha_d)/2\alpha_c$. Their analysis shows that the propagation velocities are in good agreement with available experimental data. We conclude that the effects of virtual mass introduced by [29] and [30] are necessary to describe correctly the acceleration of the dispersed phase. However, because the space-time distribution effects have been arbitrarily neglected, there remain questions about the precise form of the equations for p'_{ki}.

SAMPLE NUMERICAL PROBLEMS

A number of numerical experiments have been performed using the models described in the previous two sections, as well as simple variations of them. The first problem studied involved the acceleration from rest of a stratified steam–water mixture ($\alpha_1 = 0.5$) in a 1 m long pipe due to a constant pressure difference between the inlet and outlet. The stratified flow model was used with the wall and interface mass, heat and momentum transfer set to zero. The channel height was chosen so that the flow equations were hyperbolic only for the first 1.2 ms of the transient. A simple explicit numerical solution procedure was used with the advective terms approximated by "upwind" finite difference operators.

Figure 2 shows the steam velocity transients 0.5 m from the inlet obtained with 8, 16, 32, 64 and 128 equal size space increments and time increments chosen to maintain a constant Courant number ($a^*\Delta t/\Delta x$). The steam velocity transients obtained with 8–64 space increments were nearly identical suggesting that spatial convergence had been achieved and there was no indication of solution instability. However, when 128 space increments were used, the solution deviated from the others within two milliseconds and oscillations of rapidly increasing amplitude developed. The fact that the instability occurred after the stability condition [20] was violated is quite possibly coincidental because the complex eigenvalues are associated with the surface waves whose time scale is much longer than that of the pressure oscillations associated with the numerical instability. Nevertheless, the differential equation system is linearly unstable in this case, and the smoothness of the solutions with fewer space increments suggests that the inherent numerical damping stabilizes the solution algorithm. The instability is not simply due to round-off problems, because solutions with different arithmetic precision had the same behaviour.

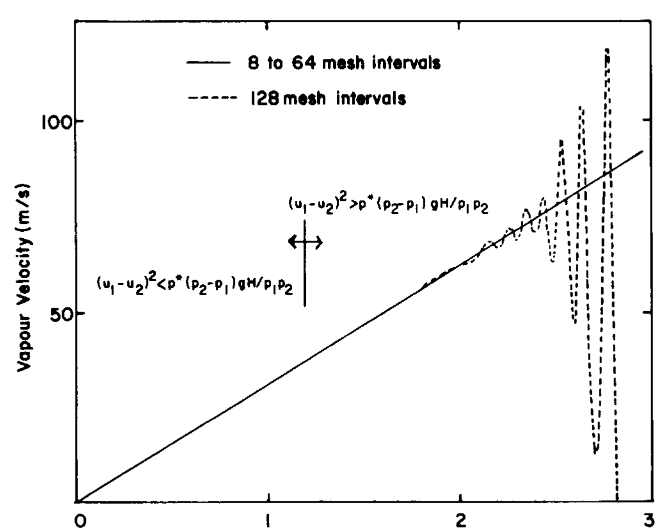

Figure 2. Vapour velocity at mid-length of a duct containing a stratified gas–liquid flow that is accelerated from rest.

A second problem studied is an idealization of the Edwards & O'Brien (1970) blowdown experiment with a 4-m long, 32-mm-dia. horizontal pipe. Both stratified and bubble flow models were used with the following wall and interface transfer terms:

$$f_{1w}c_{1w}/A = \alpha_1(0.2\,\mathrm{Re}_1^{-1/5})/d$$

$$f_{2w}c_{2w}/A = \alpha_2(64/\mathrm{Re}_2)/d$$

$$\lambda_i c_{ki}/A = 1000\alpha_1\alpha_2 c_{pk}\rho_k$$

$$f_i c_i/A = \alpha_1\alpha_2(1+75\alpha_2)/d \text{ (stratified flow)}$$

$$C_D c_i/A = \alpha_d(48/\mathrm{Re}_d) \quad \text{(bubble flow)}$$

where c_{pk} and Re_k are the specific heat and Reynolds number of phase-k respectively. Re_d is the bubble Reynolds number defined by

$$\mathrm{Re}_d = \frac{2\rho_c|u_d - u_c|R}{u_c}.$$

In the bubble-flow model the following assumptions were made: $c_p = 0$, $C_{vm} = 0.5$ and $R = 0.5$ mm. The virtual mass equation [30] was used with $\lambda_d = 0.5$, 0 and 1. However, these choices resulted in imaginary interfacial propagation velocities before choked flow was established at the outlet. In the results that follow we used a virtual mass term of the following form:

$$C_{vm}\alpha_d\rho_c\left[\frac{\partial}{\partial t}(u_d - u_c) + u_v\frac{\partial}{\partial x}(u_d - u_c)\right]$$

where $u_v = (\alpha_d\rho_d u_d + \alpha_c\rho_c u_c)/\rho^*$. This form resulted in a hyperbolic equation set for the complete blowdown.

The mathematically correct treatment of the outflow boundary condition (for hyperbolic models) is clear; in the case of single-velocity models (e.g. HEM, EVUT), we specify the outlet pressure and use the compatibility equations for the inward-directed characteristic to calculate the velocity. If the resulting velocity is greater than the speed of sound, we instead specify $u_m^* = a_m^*$ and use the compatiblity equation to calculate the pressure, which will be higher than the reservoir pressure. This approach requires that we know the compatibility equation, and strongly suggests that u_m^* is more appropriate than u_d and u_c, at least at the boundaries. Unfortunately, except for special cases, we do not have an exact analytic expression for u_m^* and the correct form of the compatibility equation in question is unknown. Incorrect treatment of the outflow boundary has a strong effect on the overall solution in a blowdown problem, since the total mass inventory depends on the discharge rate. In this study the eigenvalues of $|B\text{-}\lambda A|$ were calculated numerically at the outlet and the boundary condition $u_m^* = a_m^*$ was applied when appropriate.

The pressure histories 0.2 m from the closed end during the early stages were virtually identical for the stratified and bubble flow models as shown in figure 3. Also shown are results obtained using the EVUT model and homogeneous equilibrium model (HEM). The important feature is the undershoot in pressure, observed in the experiment and predicted by the models that allow for thermal non-equilibrium between the phases. The algorithm used to solve the EVUT equations differed from the others and is more dissipative; therefore, we belive that the difference between the EVUT solution and the solutions obtained from the separated and bubble flow models to be due to the solution algorithm rather than effects of unequal phase velocities.

Figure 4 shows the outlet velocities of the dispersed and continuous phases predicted by the bubble flow model as well as the mixture velocity predicted by the homogeneous equilibrium

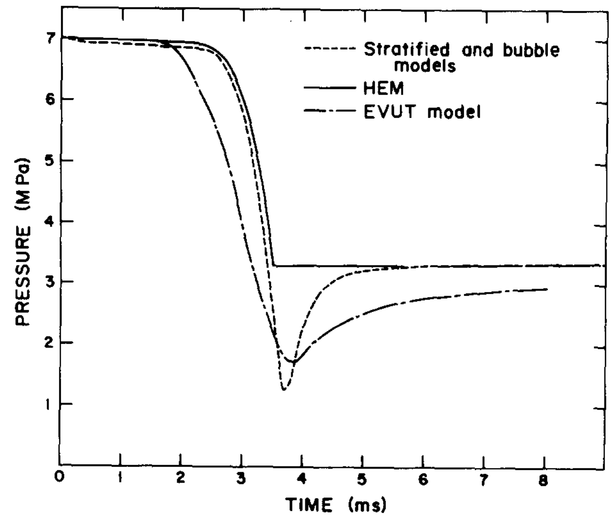

Figure 3. Predicted pressure histories near the closed end of a duct during blowdown: short term behaviour.

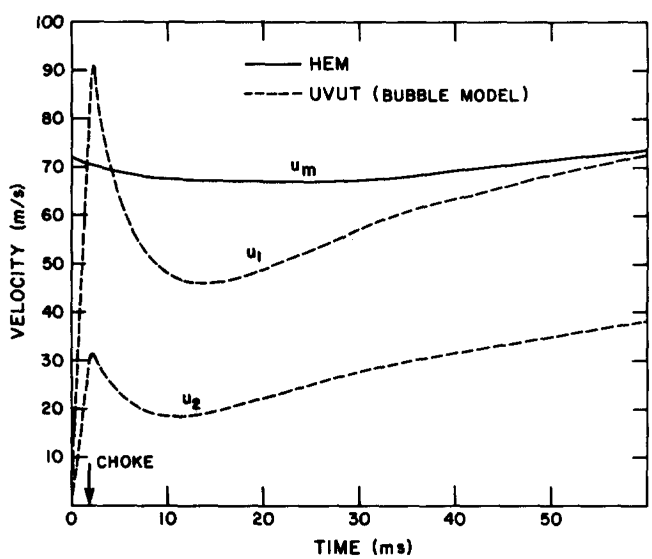

Figure 4. Gas and liquid velocity histories at the open end of a duct during blowdown: bubble flow model.

model. Not that the homogeneous equilibrium model predicts that the flow chokes almost immediately after break initiation, while the bubble flow model predicts a delay of about 2 ms. As can be seen, the choking condition $u_m^* = a_m^*$ means that $u_1 \ll a_1$ and $u_2 \ll a_2$. Following choking, the pressure at the outlet rises and, because the pressure gradient driving the flow has momentarily decreased, the gas and liquid velocities also decrease. Beyond 15 ms the flow again accelerates as vapour generation progresses within the pipe.

Figure 5 shows predicted pressure histories 0.2 m from the closed end later in the blowdown. As might be expected, pressure histories predicted by the bubble flow model using closure equations appropriate to a near homogeneous flow are in good agreement with those obtained from the homogeneous equilibrium model.

CONCLUSIONS AND RECOMMENDATIONS

We have presented one-dimensional models for simple separated (stratified, annular) and mixed (bubble, droplet) gas-liquid flows which are physically reasonable. A summary of the various constitutive terms involved in these models appears in table 1. While more detailed checks against experiments are required, the various propagation velocities are in good

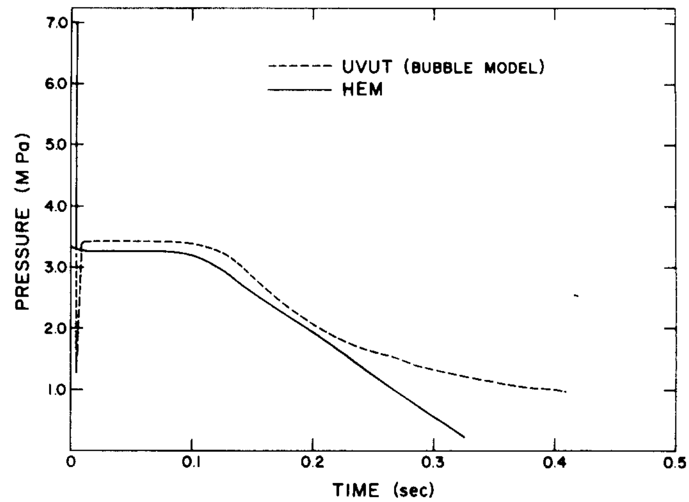

Figure 5. Predicted pressure histories near the closed end of a duct during blowdown: long term behaviour.

Table 1. Constitutive relationships

	Separated		Mixed Flows
Variables	Stratified	Annular	Bubbly, Droplet
phase-to-interface pressure difference $(p_k - p_i)$	$(-1)^k \alpha_k \rho k \frac{gH}{2}$	$\alpha_k \rho_k C_{pk} \frac{(u_1-u_2)^2}{2}$	$C_p \rho_z (u_1-u_2)^2$
phase-to-interface transfers $(\tau_{1i}, q_{1i}, q_{2i})$	$q_{ki} = \lambda_{ki} c_i (T(p_i)-T_k)/A$ $\tau_{1i} = 1/2\, f_i c_i \rho_i \lvert u_1-u_2 \rvert (u_1-u_2)/A$		
phase-to-wall transfers (τ_{kw}, q_{kw})	$q_{kw} = \lambda_{kw} c_{kw} (T_w - T_k)/A$ $\tau_{kw} = 1/2\, f_{kw} c_{kw} \rho_k u_k^2/A$		$\tau_{cw} = 2f\rho_c u_c^2/d^2$ $q_{cw} = 4\lambda_{cw}(T_w-T_c)/d$
space time distribution parameters $(\Delta_{jk}$'s)	constant		(eqn. 30) $0 \leqslant C_{vm} \leqslant 0.5$ $0 \leqslant \lambda_d \leqslant 2$
interface variables p_i, u_{ki}, h_{ki}	$p''_{ki} = p'_{ki} = 0,\ u_{ki} = u_2,\ h_{ki} = h_k(p_i)$		$p = p_i = p_k$ $u_{ki} = u_d$

agreement with experimental observations. We believe that these models provide a basis for developing a more general model for transient boiling flows. More specifically, future work should be focused on developing more general equations for the closure terms $(p_k - p_i)$, p''_{ki}, p'_{ki} and Δ_{jk}.

For a simple stratified flow (small amplitude interfacial waves), a self-consistent model has been derived assuming that the interfacial pressure and space-time distribution parameters (Δ_{jk}) are constant. The model is shown to be valid when the interface stability condition [20] is satisfied; failure to satisfy this condition indicates onset of large amplitude interfacial waves that violate the basic assumptions of the model. These waves introduce important multi-dimensional flow effects that must be accounted for in the closure equations for Δ_{jk} (Δ_{jk} can no longer reasonably be assumed constant) and p'_{ki}. Henry (1971) has suggested that virtual-mass-like terms should be added to the momentum equations. As shown for mixed flows, these terms increase the coupling between the phases as required. In our future work, we plan to use potential flow solutions for simple wave trains to determine the appropriate form of the equations for Δ_{jk} and p'_{ki}.

In the case of one-dimensional models for mixed flows, it is not clear whether the virtual-mass effects represented by [29] and [30] have been included in a self-consistent manner, because the space-time distribution effects have been neglected. The choice Δ_{jk} = constant, while convenient, is arbitrary. It is widely accepted that distribution effects must be incorporated in the mass conservation equations (e.g. through Zuber's distribution parameter C_0) to predict correctly the propagation of void disturbances (kinematic waves). Therefore there is no *a priori* reason to assume Δ_{jk} = const. Because these terms appear as derivatives, they will affect the propagation velocities of acoustic and interfacial waves. As in the case of wavy stratified flows, it may be possible to determine the relative importance and form of the Δ_{jk} and p'_{ki} terms from multidimensional potential flow solutions for idealized bubble flows.

As a consequence of the space-time averaging procedure, the interface is not explicity treated in the one-dimensional model. Therefore, external criteria are needed to determine the flow regime and basic information about the flow structure (wave amplitude, degree of entrainment, bubble size distribution, etc.). Specification of the closure equations must be based on this information.

Finally, we believe that hyperbolic flow equations, while not necessarily essential, are desirable because they have well understood mathematical properties; of particular importance are the explicit rules for boundary conditions specification and the well established numerical solution procedures. Therefore, hyperbolic systems of equations should be sought until a violation of the flow physics demands otherwise.

REFERENCES

AGEE, L. J., BANERJEE, S., DUFFEY, R. B. & HUGHES, E. D. 1978 Some aspects of two-fluid models for two-phase flow and their numerical solution. *Proc. 2nd OECD/CSNI Specialists Meeting on Transient Two-Phase Flow*, Paris.

ARDRON, K. H. & DUFFEY R. B. 1978 *Acoustic wave propagation in a flowing liquid-vapour mixture. Int. J. Multiphase Flow* 4, 303–322.

BANERJEE, S. FERCH, R. L., MATHERS W. G. & MCDONALD B. H. 1978 The dynamics of two-phase flow in a duct. *Proc. 6th Int., Heat Trans. Conf.*, Toronto.

DELHAYE, J. M. 1977 *Two-Phase Flows and Heat Transfer* (Edited by KAKAC S. & VEZIROGLU, T. N.). Vol. 1, pp. 59–114. Hemisphere, Washington.

DREW, D. A. & LAHEY, R. T. Application of general constitutive principles to derivation of multi-dimensional two-phase flow equations. *Int. J. Multiphase Flow* 5, 243–264.

DUKLER, A. E. 1978 Modelling two-phase flow and heat transfer. *Proc. 6th Int. Heat Transf. Conf.*, Toronto.

EDWARDS, A. R. & O'BRIEN, 1970 Studies of phenomena connected with the depressurization of water reactors. *J. Br. Nucl. Energy Soc.* 9, 125–135.

FERCH, R. L. 1979 Method of characteristics solutions for non-equilibrium transient flow-boiling. *Int. J. Multiphase Flow* To be published.

HANCOX, W. T., MATHERS W. G. & KAWA, D. 1978 Analysis of transient flow-boiling: Application of the method of characteristics. *A.I.C.H.E. Symp. Series* 174 **74**, 175–183.

HENRY, R. E. 1971 Pressure wave propagation through annular and mist flows. *Chem. Engng Prog. Symp. Series* **113**, 38–47.

HUGHES, E. D., LYCZKOWSKI, R. W., MCFADDEN, J. H. & NIEDERAUER, G. F. 1976 An evaluation of state-of-the-art two-velocity two-phase flow models and their applicability to nuclear reactor transient analysis. EPRI NP-143.

ISHII, M. 1976 *Thermo-fluid Dynamic Theory of Two-Phase Flow*. Eyrolles, Paris.

MATHERS, W. G., FERCH, R. L., HANCOX, W. T. & MCDOANLD, B. H. 1978 Equations for transient flow-boiling in a duct. *Proc. 2nd OECD/CSNI Specialists Meeting on Transient Two-Phase Flow*, Paris.

MORIOKA, S. & MATSUI, G. 1975 Pressure-wave propagation through a separated gas–liquid layer in a duct. *J. Fluid Mech.* **70**, 721–731.

ROUSSEAU, J. C. & FERCH, R. L. A note on two-phase separated flow models. *Int. J. Multiphase Flow* **5**, 489–494.

STANDART, G. 1964 The mass, momentum and energy equations for heterogeneous flow systems. *Chem. Engng Sci.* **19**, 227–236.

STUHMILLER, J. H. 1977 The infiuence of interfacial pressure forces on the character of two-phase flow models equations. *Int. J. Multiphase Flow* **3**, 551–560.

VERNIER, P. & DELHAYE, J. M. 1968 General two-phase flow equations applied to the thermo-hydrodynamics of boiling nuclear reactors. *Energie Primarie* **4**, 5–36.

WALLIS, G. B. 1964 *One-Dimensional Two-Phase Flow.* McGraw-Hill, New York.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves.* Wiley, New York.